On definable groups in expansions of topological fields by a generic derivation.

Françoise Point (UMons)

June 23^{rd} 2022.

Joint work with Kobi Peterzil, Anand Pillay.

Topological fields of characteristic zero endowed with a generic derivation.

There are two main directions in studying the model theory of differential topological fields (of characteristic zero), depending whether one imposes a compatibility between the derivative δ and the topology.

M. Rosenlicht tried to model what happens in Hardy fields, namely differential subfields of the field of germs at ∞ of infinitely differentiable continuous real-valued functions and recently M. Aschenbrenner, L. van den Dries et J. van der Hoeven have succeeded in axiomatizing the class of maximal Hardy fields.

We assume throughout that the derivative does not have an a priori interaction with the topology. Starting with an elementary class of large fields which is model-complete, M. Tressl has axiomatized the existentially closed (e.c.) differential expansions, assuming that the topology is definable in the field language. In these e.c. models, the derivation has a *generic* behaviour.

Here we will show how to associate to an definable group in such existentially closed (e.c.) differential expansion, a definable group in the theory we started with. We will mainly work in the finite-dimensional case.

First we will describe a class of $\mathcal{L}$-theories of topological fields amenable to this study (joint work with Pablo Cubides and Nicolas Guzy (at different times)) and

a version for these topological fields of the Weil pre-group result.

Then, we will recall some transfer results to such e.c. differential expansions and finally focus on $\mathcal{L} \cup \{\delta$-definable groups.

Framework: Topological fields (of characteristic 0)

Let T be a complete $\mathcal{L}$-theory of topological fields of characteristic 0 endowed with a definable field topology. (The topology will always assumed to be Hausdorff and non-discrete).

The language $\mathcal{L}$ contains the field language $\{+, -, \cdot, {}^{-1}, 0, 1\}$ ($\mathcal{L}_{fields}$).

We will mainly concentrate on open theories T (I) as defined in (Cubides-P/ Guzy-P)) the main hypothesis being that definable sets (on the field sort) are finite unions of Zariski closed sets intersected with definable open sets and try to point out the relationships with another set-up due to (Fornasiero-Kaplan): T is a model-complete o-minimal theory (II).

Framework: Topological fields (of characteristic 0)

In models of T, the following has been shown to hold [Cubides-P]:

1. the model theoretic algebraic closure acl satisfies the exchange property, T eliminates $\exists^\infty$ (T is a geometric theory),
2. the topological dimension coincides with the dimension induced by the model-theoretic algebraic closure acl
3. a cell decomposition theorem.

Examples–all fields are of characteristic 0

(0) Let $\mathcal{L}_< := \mathcal{L}_f \cup \{<\}$, then $RCF = Th(\mathbb{R})$ admits elimination of quantifiers in $\mathcal{L}_<$ (Tarski).

Let div be a binary relation $div(x, y)$ iff $v(x) \leq v(y)$, where v is a valuation.

1. Let $\mathcal{L}_{div} := \mathcal{L}_f \cup \{div\}$, then $ACVF$ admits elimination of quantifiers in $\mathcal{L}_{div}$ (Robinson). Ex. $ACVF_{(0,0)} = Th(\mathbb{C}((\mathbb{Q})))$,
2. Let $\mathcal{L}_d := \mathcal{L}_f \cup \{div, c_1, \cdots, c_d, P_n;\ n \geq 1\}$, where $P_n(x) \leftrightarrow \exists y\ y^n = x$, then the theory ${}_pCF_d$ of p-adically closed fields of rank d admits elimination of quantifiers in $\mathcal{L}_d$ (Macintyre, Prestel-Roquette)– ${}_pCF_1 = Th(\mathbb{Q}_p)$,
3. Let $\mathcal{L}_{<,div} := \mathcal{L}_< \cup \{div\}$, then $RCVF = Th(\mathbb{R}((\mathbb{Q})))$ admits elimination of quantifiers in $\mathcal{L}_{<,div}$ (Cherlin-Dickmann).

All the above theories of fields satisfy some *minimality* hypotheses, such as o-minimal, C-minimal, p-minimal, weakly o-minimal; more generally, dp-minimal: "no two dimensional definable grids".

Theorem (W. Johnson, 2020)

Let $(K, +, \cdot, \dots)$ be a field (no longer assumed to be a topological field) and not strongly minimal, of finite dp-rank. Then $\mathcal{K}$ can be endowed with a definable field topology (with as basis of neighbourhoods of 0, subsets of the form $X - X$, where X ranges over the definable subsets of K of maximal dp-rank (canonical topology)). Further there is a definable (field) topology which is a V-topology (*), namely induced either by an ordering or a non-trivial valuation.

(*) namely the product of two bounded sets away from 0 is away from 0. This entails that this topology is induced by either an absolute value or a non trivial valuation.)

(II) $\mathbb{R}_{an}$ the field of reals endowed with all restricted analytic functions (Denef-van den Dries),

$(\mathbb{R}, exp)$ (Wilkie),

$(\mathbb{R}_{an}, exp)$ (van den Dries, Macintyre, Marker).

Examples where we need extra-sorts (besides the field sort)

(2) Valued fields such as: $\mathbb{C}((t))$, $\mathbb{R}((t))$ and more generally the Hahn fields $k((t^\Gamma))$, where k is a field of characteristic 0 and Γ a totally ordered abelian group. In these cases, one uses the multi-sorted language $\mathcal{L}_{RV}$ and the following result of J. Flenner, building on former works of V. Weispfenning, S. Basarab and S. Basarab-F.V. Kulhmann.

Theorem (J. Flenner, 2011)

Let $\mathcal{K}$ be an henselian valued field of characteristic 0. Then the $\mathcal{L}_{RV}$-theory of $\mathcal{K}$ is an open $\mathcal{L}_{RV}$-theory.

Dimension on definable sets (in models of T

From now on, T will denote an open theory of topological fields/ or an o-minimal expansion of RCF and we restrict ourselves to definable subsets on the field sort (in the multi-sorted case)). In both cases, T eliminates the quantifier $\exists^\infty$ (namely there is a bound on the size of a uniformly definable family of finite sets).

• Topological dimension (t-dim):
Let $X \subseteq K^n$, then t-$dim(X) := max\{\ell :$ there is a projection $\pi : K^n \to K^\ell$ such that $\pi(X)$ has a non-empty interior $\}$.

• acl-dimension (acl-dim)
Let X be a definable subset of $\mathcal{K}^n$.
First define, acl-$dim(\bar{u}/A) := min\{\ell :$ there is a sub-tuple $\bar{d}$ of $\bar{u}$ of length ℓ such that $\bar{u} \in acl(A, \bar{d})\}$.
Then acl-$dim(X/A) := max\{$acl-$dim(\bar{u}/A) : \bar{u} \in X\}$.

One shows that in a model of T, these two dimensions coincide and we use dim for any of these.

Properties of the dimension in models of T

Let $X, X_1, X_2 \subset K^n$ be definable:

(D1) $\dim(X) = 0$ iff X is finite and non-empty;

(D2) $\dim(X_1 \cup X_2) = \max(\dim(X_1), \dim(X_2))$;

(D3) $\dim(Fr(X)) < \dim(X) = \dim(\overline{X})$; where $Fr(X) := \overline{X} \setminus X$ and $\overline{X}$ denotes the closure of X.

Furthermore $\dim$ is a fibered dimension, namely for every definable $X \subset K^{n+1}$:

1. $X(d) := \{a \in K^n \colon \dim(X_a) = d\}$ is definable and,
2. $\dim(\bigcup_{a \in X(d)} X_a) = \dim(X(d)) + d$.

To show these properties, one can use former results of van den Dries on dimension in models eliminating the quantifier $\exists^\infty$.

So $\dim_{\mathrm{acl}}$ defines a fibered dimension.

Comparison with properties of the dp-rank in a NIP theory

[Simon-Walsberg, 2019]: In a dp-minimal (topological) field $\mathcal{K}$, dp-rank coincides with topological dimension,
$\rightsquigarrow$ a cell decomposition theorem.

Properties of the dp-rank on definable non-empty sets in a NIP theory:

1. the dp-rank$(X) = 0$ iff X is finite,
2. if f is a definable function and X a definable set, then dp-rank$(X) \geq dp - rank(f(X))$, and dp-rank$(X) =$ dp-rank$(f(X))$ if the fibers are finite,
3. dp-rank$(X \cup Y) =$ max{dp-rank(X),dp-rank(Y)},
4. dp-rank$(X_1 \times X_2) =$dp-rank$(X_1)+$dp-rank(X_2),
5. the dp-rank is subadditive: dp-rank$(ab/A) \leq$ dp-rank$(a/Ab)+$dp-rank(b/A).

Correspondences–the Simon-Walsberg cell decomposition theorem

[Simon-Walsberg, 2019]: In a dp-minimal (topological) field $\mathcal{K}$, one has a cell decomposition theorem.

Definition

A correspondence $f : K^n \rightrightarrows K^\ell$ consists of a non-empty definable set $E \subset K^n$ together with a definable subset $graph(f) \subset K^n \times K^\ell$ such that

$$0 < |\{y \in F : \ (x, y) \in graph(f)\}| < \infty, \text{for all } \ x \in E.$$

A correspondence f is a m-correspondence if for every $x \in E$, $|\{y \in K^\ell : \ (x, y) \in graph(f)\}| = m$.

A definable subset $Y \subseteq E$ is large if $\dim(E) = \dim(Y)$ and $\dim(E \setminus Y) < \dim(E)$.

Cell decomposition

Proposition (Cubides-P)

Let $\mathcal{K} \models T$ and let E be a definable open subset of K^n. Then a correspondence $f : E \rightrightarrows K^\ell$ is continuous on a large subset of E.

A cell is the graph of a continuous m-correspondence $f : E \to K^\ell$, for some $m \geq 1$, with E a definable open set. In models of T it has the additional property of being closed under the projection under initial subsets of coordinates. To encompass open sets and points, one make the convention that when $\ell = 0$, the graph of f is E and when $n = 0$, it is a finite subset of K^ℓ.

Theorem (Cubides-P)

Let T be an open $\mathcal{L}$-theory of topological fields and let $\mathcal{K}$ be a model of T. Let X be an A-definable subset of K^{n+1}, $n \geq 0$, then X is a finite disjoint union of A-definable cells.

Generic bijections

Let T be a (complete) geometric theory.

Let $\mathcal{M} \models T$, $\mathcal{M}$ sufficiently saturated. Let A be an $\mathcal{L}$-definable subset of $\mathcal{M}$ with $A \subset M^n$ for some $n \geq 1$ and let $f : A \mapsto A$ be a definable map. Then we say that f is a generic bijection if the domain of f and the image of f are large subsets of A and on its domain, f is injective.

We say that two generic bijections on A are equal ($\sim$) if they coincide on a large subset of A.

Denote by $(Perm_g(A), \circ)$ the group of all generic bijections of A endowed with composition.

Weil pre-group construction

Proposition

Let $\tilde{G}$ be a definable subset of $\mathcal{M}$, $\tilde{G} \subset M^s$. Let Z_2 be a large definable subset of $\tilde{G} \times \tilde{G}$ and Z_3 be a large definable subset of $\tilde{G}$. Assume we have $\mathcal{L}$-definable functions $F(x, y) : Z_2 \to \tilde{G}$ and $F_{-1} : Z_3 \to \tilde{G}$ satisfying the following properties:

1. for $a \in Z_3$ we have $F_{-1}(a) \in Z_3$ and $F_{-1}(F_{-1}(a)) = a$,
2. for $a \in Z_3$, let

$$X_a := \{y \in \tilde{G} : (a, y) \in Z_2 \;\&\; (F_{-1}(a), F(y, a)) \in Z_2 \;\& \\ F(F_{-1}(a), F(a, y)) = y\},$$

   then X_a is large in $\tilde{G}$,
3. the set $\{x \in \tilde{G} : \forall a \in \tilde{G}, \text{ if } a \text{ generic over } x, \text{ then } x \in X_a\}$ is large in $\tilde{G}$,
4. for any $z \in \tilde{G}$, there is a large subset $Y(z)$ of Z_2 such that if $(x, y) \in Y(z)$, then $F(x, F(y, z)) = F(F(x, y), z)$.

Proposition (continued)

with corresponding statements on the right for items (2) and (3).

Then, letting for $c \in Z_3$, $f_c : Z_3 \to \tilde{G} : x \mapsto F(c,x)$, we have $\{f_b f_a \colon a, b \in Z_3\}$ is a subgroup of $Perm_g(\tilde{G})$ and using that $\tilde{G}$ is finite-dimensional, we endow Z_3 with the structure of a definable group.

We show that the map $c \in Z_3 \mapsto f_c$ is injective.
Let $d := \dim(\tilde{G})$ and let $t_1, \ldots, t_{2d+1}$ be $2d+1$ generic independent elements in $\tilde{G}$. Then we show that

$$\{f_d f_a \colon a, d \in Z_3 \text{ with } d \text{ generic over } a\} \subset$$

$$\{f_b f_a \colon a, b \in Z_3\} \subset \bigcup_{i=1}^{2d+1} \{f_{F_{-1}(t_i)} f_{F(t_i,b)} \colon b \in Z_3, t_i \text{ generic over } b\} \subset$$

$$\{f_d f_a \colon a, d \in Z_3 \text{ with } d \text{ generic over } a\} =$$

$$\{f_{F(d,a)} : a, d \in Z_3 \text{ with } d \text{ generic over } a\} = \{f_b \colon b \in Z_3\}.$$

Differential expansions

Let T be an $\mathcal{L}$-theory as before. We consider T_δ the theory of differential expansions of models of T by a derivation δ, namely $T_\delta := T \cup \{\delta(x+y) = \delta(x) + \delta(y),\ \delta(xy) = x\delta(y) + \delta(x)y\}$.

Note that:

- we impose no conditions on δ with respect to the topology.

- in models $\mathcal{K}$ of T_δ, we always get a pair of fields (K, C_K), where $C_K := \{x \in K : \delta(x) = 0\}$.

Existentially closed differential expansions

In both cases, one axiomatizes the existentially closed models of T_δ, possibly under some additional conditions on T.

For $T = RCF$, it was done in 1978 by M. Singer, one gets the theory of closed ordered differential fields ($CODF$).

In case (I), if T admits Q.E., then the theory T^*_δ of existentially closed models of T_δ is given by T_δ together with the following scheme of axioms (DL):

for $\mathcal{K} \models T_\delta$, for every differential polynomial $P(x) \in K\{x\}$ with $\ell(x) = 1$ and $\mathrm{ord}_x(P) = m \geqslant 1$, for $y = (y_0, \ldots, y_m)$, we have that for any neighbourhood W of 0 in K^{m+1},

$$(\forall y)\big((P^*(y) = 0 \wedge s^*_P(y) \neq 0) \rightarrow \exists x \\ \big(P(x) = 0 \wedge s_P(x) \neq 0 \wedge (\bar{\delta}^m(x) - y) \in W, \big)\big)$$

where $s^*_P := \partial_{y_n} P^*$ and $s_P := \partial_{\delta^m(x)} P(x)$.

Existentially closed differential expansions

In case (II) (Fornasiero-Kaplan), the derivation δ is assumed to be a T-derivation, namely δ is compatible with any C^1-0-definable function $f : U \to K$, where U is an open subset of K^n and $\mathcal{K} \models T$, i.e.

$$\delta(f(\bar{u})) = J_f(\bar{u})\delta(\bar{u}),$$

where $J_f := (\partial_{x_1} f, \ldots, \partial_{x_n} f)$.

Then, the theory $T_{\delta,g}$ of existentially closed models of T_δ is given by T_δ together with:

given $\mathcal{K} \models T_\delta$, we have for every $m \geq 1$ and every $\mathcal{L}(K)$-definable set $A \subset K^{m+1}$ if $\dim(\pi_m(A)) = m$, then there is a differential tuple in A, where π_m is the projection on the first m coordinates.

In models of $T_{\delta,g}$, one says that δ is T-generic.

Generic expansions by a derivation in open $\mathcal{L}$-theories

In case (I), to get consistency of T^*_δ, we work in a class of large fields, a notion due to F. Pop.

[Pop, 1996] A field K is large if every smooth curve over K has infinitely many K-points, provided it has at least one. Equivalently K is large if K is existentially closed in $K((t))$. Examples of large fields: PRC, PAC, real-closed fields, p-adically closed fields.

In our setting, we require that we can find an elementary extension K^* of K with $K \subset K((t)) \subset K^*$ where $tK[[t]]$ are sent to small elements of K^* with respect to K.

Transfer results from T and T^*_δ: QE, NIP, existence of a fibered dimension function, elimination of $\exists^\infty$, distality, EI (using the open core property), a $\mathcal{L}_\delta$-cell decomposition theorem.

Furthermore we can apply our results to theories of dense pairs of models of T since if $K \models T^*_\delta$, then $C_K \models T$.

Prolongations in models of T_δ^*

Let $\mathcal{M} \models T_\delta^*$ and let $X \subset M^n$ be an $\mathcal{L}_\delta$-definable subset of $\mathcal{M}$. We consider the m-prolongation of $a = (a_1, \ldots, a_n) \in X$ as the $n(m+1)$-tuple $a^{\nabla_m} := (a, \bar{\delta}(a), \bar{\delta}^2(a), \ldots, \bar{\delta}^m(a))$, where $\bar{\delta}(a) = (\delta(a_1), \ldots, \delta(a_n))$.
The set of m-prolongations coming from elements of X, is denoted by $X^{\nabla_m} := \{a^{\nabla_m} : a \in X\}$.

We also consider the infinite prolongation of $a = (a_1, \ldots, a_n) \in X$ as the infinite tuple $a^{\nabla_\infty} := (a, \bar{\delta}(a), \bar{\delta}^2(a), \ldots)$; we call this infinite tuple differential thread.
The set of differential threads coming from elements of X, is denoted by $X^{\nabla_\infty} := \{a^{\nabla_\infty} : a \in X\}$.

δ-compatible correspondences in models of T-case (I)

Let $\mathcal{K}$ be a model of T_δ^* and let $f : U \rightrightarrows K$ be a $\mathcal{L}$-definable correspondence, where U is an open subset of K^n. Let $a := (a_1, \ldots, a_n) \in U$.

Definition

Then f is compatible if $\delta(f(a)) = g(a, \delta(a))$, for some $\mathcal{L}$-definable correspondence $g : U \to K$.

When f is a correspondence from K^n to K^m, we compose f with the projections $\pi_i : K^m \to K$, $1 \leq i \leq m$ and define f compatible with δ if each of the functions $f \circ \pi_i$ is compatible with δ.

Proposition

Let $\mathcal{K}$ be a model of T and U an open definable set in K^n. Then a $\mathcal{L}$-definable correspondence $f : U \rightrightarrows K^m$ with $\mathcal{K}$ a model of T is compatible.

Dimension in models of T_δ^*

Let $\mathcal{K} \models T_\delta^*$ and from now on we will assume that $\mathcal{K}$ is sufficiently saturated. Let $X \subset K^n$ be an $\mathcal{L}_\delta(k)$-definable subset of $\mathcal{K}$, k a differential subfield of K.

We say that X is finite-dimensional (f.d.) if
$\max_{a \in X}$ $trdeg_k$ $k(a^{\nabla_\infty})$ is finite,
more generally, if $\max_{a \in X} \dim_k(k(a^{\nabla_\infty}))$ is finite.
In this case, set $\dim_\delta(X) := \max_{a \in X} \dim_k(k(a^{\nabla_\infty}))$.
A tuple $a \in X$ is generic if $\dim_k(k(a^{\nabla_\infty})) = \dim_\delta(X)$.

If X is f.d., it implies that there is m such that for every $a = (a_1, \ldots, a_n) \in X$, $\delta^{m+1}(a_i) = f_i(\bar{\delta}^m(a))$, $1 \leq i \leq n$, for some $\mathcal{L}(k)$-definable correspondence. Since f_i is compatible, we can express the successive derivatives of $\delta^m(a_i)$ in terms of $\bar{\delta}^m(a)$.

Definable subsets in models of $T_\delta^*/T_{g,\delta}$

Let $X \subset K^n$ be an $\mathcal{L}_\delta(k)$-definable subset of $\mathcal{K}$, k a differential subfield of K.

Then $X = \nabla^{-m}(Z)$ for some m and some $\mathcal{L}(k)$-definable set Z.

In case of T_δ^* one uses the transfer of QE from T to T_δ^*, going to the Morleysation of T.
In case of $T_{\mathcal{G},\delta}$, one uses the axiomatisation and cell-decomposition theorem.

Proposition

In models $\mathcal{K}$ of $T^*_\delta /\ T_{\delta,g}$, we have: $\mathrm{acl}_\delta = \mathrm{acl}$. More precisely let $a \in \mathrm{acl}_\delta(\langle X\rangle)$ where $\langle X\rangle \subset K$ is the $\mathcal{L}$-substructure generated by X, $a \in K$. Then there is $m \geq 0$ such that $a \in \mathrm{acl}(\nabla_m(\langle X\rangle))$.

Therefore, given $f : X \to K$ be an $\mathcal{L}_\delta$-definable correspondence, with X an $\mathcal{L}_\delta$ definable set of K^n, we can find $m \geq 0$ and an $\mathcal{L}$-definable correspondence $F : Y \to K$ with $Y \supset X^{\nabla_m}$ an $\mathcal{L}$ definable set of $K^{n(m+1)}$ such that f and F agree on differential tuples.

Sections

Let $X \subset K^n$ be an $\mathcal{L}_\delta(k)$-definable subset of $\mathcal{K}$, k a differential subfield of K. Let $X = \nabla_m^{-1}(Y)$ for some $\mathcal{L}$-definable set Y.
Assume now that X is f.d. for each $a \in X$, there is m such that $\delta^{m+1}(a) = g(\bar{\delta}^m(a))$, with g is an $\mathcal{L}$-definable function depending on a, but defined on an open set containing a. Using a compacity argument, we get finitely many such $\mathcal{L}$-definable g_j, $j \in J$, defined on Y_j.
So we define a section s_j sending $u := (u_0, \ldots, u_m) \in Y$ to $(u, (u_1, \ldots, u_m, g_j(u_0, \ldots, u_m)) \rightsquigarrow (Y, s)$.
Finally let $f : X \to K^\ell$ be an $\mathcal{L}_\delta$-definable correspondence.
We get a finite covering of Y and on each piece, we may associate with f an $\mathcal{L}$-definable correspondence F which agrees with f on differential tuples (coming from X).
Moreover $\nabla_m(X)$ embeds in the subset of u in Y such that $(u, \delta(u)) = s(u)$ for some s an $\mathcal{L}$-definable map.

Let $\mathcal{K} \models T_\delta^*$, and assume that $\mathcal{K}$ is $|k|^+$-saturated, where k is a differential subfield.

Let $\Gamma := (\Gamma, \times, {}^{-1}, 1)$ be an $\mathcal{L}_\delta(k)$-definable group in $\mathcal{K}$ and assume that Γ is finite-dimensional. Then there is an $\mathcal{L}$-definable subset Y and $m \geq 0$ such that $\nabla^{-m}(Y) = \Gamma$ and $\mathcal{L}$-definable functions F and F_{-1} on large definable subsets of respectively $Z_2 \subset Y \times Y$ and $Z_3 \subset Y$ containing $\nabla^m(\Gamma)$ and F, respectively F_{-1}, coincides with $f_\times$ respectively f_{-1} on differential tuples, with the following properties:

Induced structure on Y

Proposition

Under the above assumptions, (Y, F, F_{-1}) has the following properties:

1. F_{-1}^2 is the identity on Z_2,
2. for $a \in Z_2$, there is a subset $X_a \subset Y$ where the map sending x to $F(a, x)$ is well -defined and its inverse is given by the map sending y to $F(F_{-1}(a), y)$ and the set $\{a \in Z_2 \colon X_{\bar{a}}$ is a large subset of $Z_2\}$ is a large subset of Y,
3. the associativity property holds generically, namely there is a large set $V \subset Y$ such that any element $z \in V$ there is a large set $Y(z) \subset Z_3$ such that for any $(x, y) \in Y(z)$,

$$F(x, F(y, z)) = F(F(x, y), z)$$

Proposition (continued)

1. the two-sided cancellation property holds generically, namely there is a large subset $V_{canc} \subset Y$, if $x, y \in V_{canc}$, for almost every a (equivalently for any generic a over $\{x, y\}$), then

$$(F(a, x) = F(a, y)) \to (x = y)$$

$$(F(x, a) = F(y, a)) \to (x = y).$$

Corollary (Peterzil-Pillay-P)

Let $\Gamma := (\Gamma, f_\times, f_{-1})$ be a finite-dimensional $\mathcal{L}_\delta(k)$ definable group of $\mathcal{K}$. Then there is $m \geq 0$ and a prolongation $\nabla_m(\Gamma)$ which embeds in an $\mathcal{L}$-definable group with compatibility of the group operations on differential points.

D-groups and $CODF$

Let us consider $\mathcal{K} \models CODF$ and k a differential subfield of K. In this case our result takes the following form:

Proposition (Peterzil-Pillay-P)

An $\mathcal{L}_\delta(k)$ finite-dimensional group Γ definable in $\mathcal{K}$ is isomorphic to the differential points of (G, s) with (G, s) an affine Nash D-group.

By former results of Hrushovski and Pillay on groups definable in RCF, if G is semi-algebraically connected affine Nash group, then G is Nash isogeneous to the semi-algebraic connected component of the group $H(K)$, where H is an algebraic group.